Staggered-magnetization approach to spin-glasses

H. Sompolinsky

Department of Physics, Harvard University, Cambridge, Massachusetts 02138
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A method is developed to solve the Sherrington-Kirkpatrick infinite-ranged spin-glass model, considering the staggered magnetizations associated with the eigenvalue spectrum of the random interaction matrix as the spin-glass order parameters. It is shown that the spin-glass ordering is not associated with a macroscopic condensation of eigenstates, but instead each of the $N$ eigenstates acquires, below $T_c$, nonzero magnetization of $O(1)$.

In recent years there has been considerable interest in understanding the low-temperature properties of spin-glasses. Thus far, an adequate theory of the problem has not been formulated even in the mean-field limit. The failure of simple theories, like the original replica approach, indicates that the Edwards-Anderson order parameter is not sufficient for the description of the spin-glass order. Modification of the replica method requires the introduction of additional order parameters associated with the breaking of replica symmetry which, however, do not bear any clear physical interpretation. Alternative approaches such as that of Thouless, Anderson, and Palmer (TAP) and Sommers's used diagrammatic techniques which are applicable only to the Sherrington-Kirkpatrick (SK) infinite-ranged model and do not give a general qualitative picture of spin-glass ordering. In addition, Sommers's solution, though physically acceptable, differs from the results of numerical simulations of the model as well as from the recent replica symmetry-breaking solution of Parisi. Thus, it is useful to develop an alternative and more physical approach to the problem.

In this paper, the staggered spin modes associated with the eigenstates of the random exchange interaction matrix are considered as the natural spin-glass order parameters. This approach is related to a suggestion by Anderson that the spin-glass ordering is associated with the appearance of macroscopic staggered magnetization in the first extended eigenstate of the exchange interaction matrix, similar to ordinary second-order transitions. In fact, such a phase has been shown to exist in a spherical model of spin-glass with infinite-ranged interactions. Using the properties of the eigenstates of large random matrices, a method is developed to solve the SK model without averaging out the randomness at an early stage. This method is shown to reproduce in a straightforward way Sommers's solution (besides the original SK unstable solution) as the unique solution of the model. In addition, it is shown that due to the local interaction between the eigenstates, the spin-glass order is not associated with macroscopic condensation of modes but is characterized by a distribution of magnetizations throughout the whole spectrum, each of the $N$ eigenstates acquires a magnetization of $O(1)$.

The present approach can also be used to investigate other aspects of infinite-ranged spin-glasses (e.g., their dynamic properties). It may also serve as a useful starting point for formulating an adequate theory of short-ranged spin-glasses by using "eigenstate-space" techniques analogous to the momentum-space methods of ordinary phase transitions.

The SK Hamiltonian

$$H = -\frac{1}{2} \sum_{i,j} J_{ij} S_i S_j ,$$

(1)

describes $N$ Ising spins $(S_i = \pm 1)$ interacting via a random infinite-ranged interaction with a Gaussian distribution

$$P(J_{ij}) = \frac{1}{(2\pi)^{1/2}} \exp\left(-NJ_{ij}^2/2\right) .$$

(2)

Diagonalizing the matrix $J_{ij}$, the Hamiltonian is written as $H = -\sum S_i S_j$ where $S_i = \sum \langle i | S | \rangle S_i \langle i | S \rangle$ being the (real) orthonormal eigenvector of $J_{ij}$ corresponding to the eigenvalue $J_\lambda$. In the limit $N \to \infty$ the eigenvalue density obeys the semicircular law

$$\rho(J_{\lambda}) = \frac{4J^2 - J_{\lambda}^2}{2\pi J^2} , \quad |J_{\lambda}| \leq 2J .$$

(3)

Since $J_{ij}$ does not possess any spatial symmetry or length scale it is obvious that $\langle \langle i | S | \rangle \rangle_\infty = 0$. The invariance of the ensemble defined by Eq. (2) under orthogonal transformations implies that: (i) averages over products of $\langle \lambda | i \rangle$ can be done separately from integration over eigenvalue densities; (ii) averages of all correlations among products of eigenvector amplitudes of different eigenstates or different sites vanish except for the correlations implied by their orthogonality and completeness properties. These properties allow for
performing the quenched random average in the following way: Quantities are expressed as functions of eigenvalues and (strongly fluctuating) eigenvector amplitudes. First, averaging over the amplitudes $\langle \langle i | \lambda \rangle \rangle$ is performed by summing over all possible $\lambda$ and $i$ and using the orthonormality and completeness relations. Then, the remaining smooth dependence on $J_\lambda$ is integrated via Eq. (3). To illustrate the method, consider for a quantity of the type

$$ N^{-2} \sum_{\lambda \neq \nu} \sum_{j} g(\lambda) g^{(j)}(\nu) f_j (i | \lambda) \langle \langle i | \nu \rangle \rangle \langle \langle j | \lambda \rangle \rangle \langle \langle j | \nu \rangle \rangle , \tag{4} $$

where the dependence of $g$ on $\lambda$ is only through $J_\lambda$, and $f_j$ is a local random quantity. Then, according to the above rules the average of Eq. (4) is

$$ \langle \langle f_j^2 \rangle \rangle \approx \langle \langle f_j \rangle \rangle^2 \left[ \int g(J) p(J) dJ \right] \tag{5} $$

the first term is the contribution of the terms in Eq. (4) with $\lambda \neq \nu$ and the last term coming from those with $\lambda = \nu$.

Using this method to calculate diagrams in the $\lambda$ space, the correct free energy above $T_c$ (Refs. 2 and 3) as well as the TAP mean-field equations below it, were obtained,\textsuperscript{15} thus indicating that the above averaging procedure is indeed exact in the thermodynamic limit. Above $T_c$, the size of the staggered modes in triggering the transition is demonstrated by the divergence at $T_c$ of the fluctuation of the mode with the largest eigenvalue, i.e., $\langle S^2 \rangle = (1 - \beta J)^{-2}$, $T \geq T_c$\textsuperscript{3,15}

Proceeding to investigate the low-temperature phase the TAP equations\textsuperscript{8} are written in terms of $M_\lambda = \langle S_\lambda \rangle \langle \cdot \cdot \cdot \rangle$ means thermodynamic average) as

$$ M_\lambda = \sum_i \langle \langle i | \lambda \rangle \rangle \langle S_i \rangle $$

$$ = \sum_i \langle \langle i | \lambda \rangle \rangle \tanh \left[ \sum_{\nu} \langle A(J_\nu) M_\nu + \beta h_\nu \rangle \langle \langle i | \nu \rangle \rangle \right] , \tag{6} $$

where $A(J_\nu) = \beta J_\nu - \beta^2 J^2 (1 - q)$, $q$ is the Edwards-Anderson order parameter $q = \langle \langle S \rangle \rangle^2 \approx = N^{-1} \sum_i \langle \langle S_i \rangle \rangle^2$, and $h_\lambda$ are small staggered fields. Imagine that a particular $M_\nu$ is nonzero. Then since the quantities of the type $\langle \langle \langle i | \lambda \rangle \rangle \langle \langle \nu | \nu \rangle \rangle ^2 \rangle$, with $\lambda \neq \nu$, do not vanish, Eq. (6) implies that other magnetizations $M_\nu$ with $\lambda \neq \nu$ do not vanish, too. Let us first assume that each of the $M_\nu$ is of $O(1)$, namely, that no macroscopic condensation of modes occurs. In such a case, the dependence on $J_\lambda$ of the right-hand side of Eq. (6) comes, in the limit $N \to \infty$, only via the terms linear in $A(J_\lambda) M_\lambda$; hence, $M_\lambda$ can be written

$$ M_\lambda = m_\lambda + \chi_\lambda [A(J_\lambda) M_\lambda + \beta h_\lambda] , \tag{7} $$

where $m_\lambda$ depends on $\lambda$, only through $\langle \langle i | \lambda \rangle \rangle$, and therefore $\langle m_\lambda \rangle$ is independent of $\lambda$.

$\chi_\lambda = \delta(M_\lambda) / \delta(A(J_\lambda) M_\lambda)$ is a nonrandom quantity given by

$$ \chi_\lambda = 1 - q + \sum_i \langle \langle i | \lambda \rangle \rangle - (\langle \langle \langle i \rangle \rangle \rangle)^2 ) $$

$$ \times \sum_{\nu \neq \lambda} \langle \langle i | \nu \rangle \rangle A(J_\nu) \frac{\delta M_\nu}{\delta \langle A(J_\lambda) M_\lambda \rangle} . \tag{8} $$

$\chi_\lambda$ is also related to the local susceptibility by

$$ \chi = N^{-1} \sum_\lambda \frac{\delta M_\lambda}{\delta h_\lambda} = \beta \chi_0 \int \rho(J) [1 - \chi_0 A(J)]^{-1} dJ , \tag{9} $$

which follows directly from differentiating Eq. (7).

The dispersion of $\langle M_\lambda \rangle_{\nu}$ immediately follows from Eq. (7).

$$ \langle M_\lambda \rangle_{\nu} = Q / [1 - \chi_0 A(J_\lambda)]^2 , \tag{10a} $$

$$ Q = q / \int \rho(J) [1 - \chi_0 A(J)]^{-2} dJ \tag{10b} $$

Consider now the random quantity $z$ defined by

$$ (1 + \chi_0 I_1) z = \sum_\lambda \langle \langle i | \lambda \rangle \rangle m_\lambda [[A^{-1}(J_\lambda) - \chi_0]^{-1} - I_1] $$

$$ (1 + \chi_0 I_1) (\tanh^{-1}(S_i) - \beta h_i) - I_1 \langle S_i \rangle , \tag{11a} $$

$$ I_1 = \int \rho(J) [A^{-1}(J) - \chi_0]^{-1} dJ \tag{11b} $$

Since integration over $J_\lambda$ of the factor multiplying $\langle \langle i | \lambda \rangle \rangle m_\lambda$ in Eq. (11a) is identically zero (by the definition of $I_1$) the only terms which contribute to $\langle z^{2n} \rangle_{\nu}$ come from all possible pairings of $\lambda$'s. Thus $z$ is a Gaussian variable with a variance $Q^{1/2} (I_2 - I_1^2) / (1 + \chi_0 I_1)$, from which it follows that

$$ q = \int \frac{dx}{\sqrt{2 \pi}} e^{-x^2 / 2} S^2(x) , \tag{13} $$

$$ x = \beta (1 - q + \Delta) = \int \frac{dx}{\sqrt{2 \pi}} e^{-x^2 / 2} \frac{dS(x)}{dh} \tag{14} $$

where $S(x) = \tanh[Ax + BS(x) + \beta h]$ with

$$ A = \frac{Q^{1/2} (I_2 - I_1^{1/2})}{1 + \chi_0 I_1} = J \beta q^{1/2} , \tag{15a} $$

$$ B = \frac{I_1}{1 + \chi_0 I_1} - \beta^2 J^2 \Delta . \tag{15b} $$
The last equalities are identities derived by calculating the relevant integrals with \( \rho(J) \) of Eq. (3). Equations (13)–(15) reproduce exactly Sommers’s metastable\(^{18} \) solution\(^a \) (for \( \Delta \neq 0 \)) as well as the original SK unstable solution \( (\Delta = 0) \). In particular, near \( T_c \), the solution \( \langle x \rangle \) of the equation \( q = q \equiv r + \frac{1}{2} t \) \( (r = 1 - T/T_c) \), as compared to the marginally stable result\(^b \) of Parisi’s solution \( q \equiv t + r,^{17} \)

As noted by Sommers, at \( T \leq 0.6 T_c \), the function \( S(x) \) is not uniquely defined but rather consists of three branches in the region \( |x| \leq x_0, x_0 \) defined by \( dx/dS = 0 \). Selecting a particular \( S(x) \) by minimizing the averaged free energy, as Sommers does, results in a value of \( 1 - q \) which vanishes exponentially as \( T \) approaches 0 \( K \), in marked contrast to the result \( 1 - q \propto T^2 \) obtained by numerical simulation\(^1,8,10 \) of the model. It should be noted, however, that the above equations allow for other choices of \( S(x) \) as well, some of which result in metastable solutions which do obey \( 1 - q \propto T^2. \)\(^{15} \) This indicates that a full analysis of the low-temperature properties implied by Eqs. (13)–(15) may require the construction of an appropriate averaging procedure over the various metastable solutions.

Thus far, we have assumed that none of the modes acquires macroscopic magnetization. In order to investigate other possibilities let us assume that a particular mode \( \lambda_0 \) (perhaps with the largest eigenvalue) acquires magnetization \( M_0 \) of the order \( N^{1/2} \). In such a case, Eq. (7) does not hold any more for \( \lambda = \lambda_0 \). Hence one has to exclude the term with \( \lambda = \lambda_0 \) from the definition of the Gaussian variable \( z \), as in Eq. (11a). Since correlations between \( M_0 (x|\lambda_0) \) and \( z \) are only of \( O (N^{-1/2}) \) one can treat \( (x|\lambda_0) \) as an independent Gaussian variable, \( y \equiv z + \Delta \), and therefore \( q, x, \) and \( M_0 \) are given as double Gaussian integrals of \( S^3(x,y), \delta S(x,y)/\delta h \), and \( N^{1/2} y S(x,y) \), respectively, with

\[
\begin{align*}
\text{tanh}^{-1} S(x,y) &= J\beta(q - M_0^2)^{1/2} x \\
&\quad + [A(J_0) - \beta^2 J^2 \Delta]M_0 N^{-1/2} y \\
&\quad + \beta^2 J^2 \Delta S(x,y) + \beta h. 
\end{align*}
\]

Asymptotically close to \( T_c \), these integrals do allow for a solution with \( M_0 \neq 0 \) as was in fact claimed by Thouless et al.\(^8 \) However including terms of at least \( O(r^4) \), it is found that the only self-consistent solution is \( M_0 = 0 \). This method can be easily generalized to exclude the possibility of solutions with any finite number of macroscopically condensed modes.

Physically, the absence of macroscopically condensed eigenstates results from the coupling between the collective spin modes which is particularly strong in the absence of spatial symmetry associated with the modes. The existence of large numbers of strongly-coupled order parameters is also manifested in the deviation of \( x \) from linear-response theory which predicts\(^18 \) \( \Delta = 0 \). This is best seen in terms of the staggered susceptibilities \( x_0 = \partial M_0/\partial h \). These susceptibilities are given by \( N^2 \left( \Delta^2 - \Delta^3 + \frac{1}{2} J^2 (1 - q) - J_0 \beta \right) \) [see Eq. (9)], whereas the staggered fluctuations \( \beta\left( \langle S_x \rangle - \langle S_x \rangle^2 \right) \) are \( \Delta^2 \) equal to \( \left( 1 - q \right)^{1/2} + \frac{1}{2} J^2 (1 - q) - J_0 \beta \). As is seen by Eq. (8), the deviation\(^{19} \) of \( x_0 \) from \( 1 - q \) results from the fact that a small variation of a particular ordering field \( h \) affects not only the value of \( M_0 \) but also that of all other order parameters as well, presumably leading the system to a completely new ground state.\(^{20,21} \)

These features are probably characteristic of any spin-glass models with local interactions between the collective spin modes. On the other hand, Gaussian models like the spherical spin-glass model are expected to obey linear-response theory as indeed found by Kosterlitz et al.\(^{13} \) It is also interesting to note that, unlike the spherical model, the critical properties associated with the spin mode of the largest eigenvalue are very different from ordinary second-order transitions. In particular, while the critical exponent of \( x_{2J} \), above \( T_c \), is \( \gamma = 2 \), below \( T_c \) it is \( \gamma = 4. \)\(^{22} \) Thus, the critical exponent \( \beta \) associated with \( M_{2J} \) as given by Eqs. (10a) and (10b) is negative, i.e., \( \beta = -\frac{1}{2} \). The scaling law \( 2 \beta y + \alpha - 2 \) is obeyed (with \( \alpha = -1) \).\(^{22} \) Note, however, that the divergence of \( M_{2J} \) is meaningful only when compared to 1. Actually, the negative exponent implies that at \( T_c, M_{2J} \) scales as \( N^{1/2} \) and not as 1.

Finally, it should be noted that most recently there have been claims\(^{23,24} \) that the TAP equations possess a large number of solutions associated with order parameters which are missing in the SK and Sommers’s solutions. The physical meaning of these order parameters is, however, not clear, and it is doubtful whether they can be reproduced by directly solving the TAP equations.

Concluding it may be remarked that the approach outlined above can be used also to treat other aspects of infinite-ranged spin-glasses such as their dynamics above and below \( T_c. \)\(^{15} \) This approach can also be extended to construct a mean-field theory for short-ranged spin-glasses. Such a theory seems to predict a zero transition temperature for any spin dimensionality provided the density of states at the mobility edge of the (renormalized) interaction is finite. Details will be published elsewhere.\(^{15} \)

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16. Stability of the solutions with respect to $\Delta$ is determined by $1-q + 2\langle S_i \rangle \mu < T^2 / T^2$. Local stability with respect to $\langle S_i \rangle$ is determined by $dS/dx > 0$. Note, however, that Eqs. (13)–(15) are equivalent to a solution derived by a particularly simple replica symmetry-breaking scheme, see A. J. Bray and M. A. Moore, J. Phys. C 13, 419 (1980).
18. Note that $X_0 = 1 - q$ is equivalent to $\Delta = 0$, since for this value of $X_0$, $I_1$ of Eq. (12) is identically zero.
21. This holds for both the SK and Sommers's solutions, since in the latter case $X_0 - (1-q) = O(r^4)$.