Spatial Coherence and Temporal Chaos in Macroscopic Systems with Asymmetrical Couplings

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Coupled map lattices with asymmetric short-range couplings are studied analytically and numerically. It is shown that with open boundary conditions these systems exhibit spatially uniform, but temporally chaotic states that are stable even in the thermodynamic limit. The stability of this state is associated with the appearance of a gap at zero wave number in the spectrum of the linear operator describing the fluctuations about the uniform state. The long-range order is unstable to noise. We calculate the finite coherence length of the chaotic state in the presence of weak noise.

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Can a long-range spatial order exist in a temporally chaotic macroscopic system with short-range interactions? Studies of this question using a variety of models, in particular coupled map lattices (CML), have shown that in the temporally chaotic states of these models the spatial correlations decay with a finite correlation length [1–5]. These results have been interpreted to show that generally temporal chaos in large systems leads to the loss of spatial coherence. The purpose of this Letter is to show that stable long-range coherence can coexist with temporal chaos even in macroscopic systems with short-range interactions (see also discussions in [6]).

We consider a one-dimensional CML [3,7,8] with nearest-neighbor couplings, given by the following equations:

\[
x_j(n+1) = (1 - \gamma_1 - \gamma_2)f(x_j(n)) + \gamma_1 f(x_{j-1}(n)) + \gamma_2 f(x_{j+1}(n)),
\]

where \(\gamma_1\) and \(\gamma_2\) are the coupling constants, and \(f(x)\) is a map of the interval onto itself. For all values of \(\gamma_1\) and \(\gamma_2\), Eqs. (1) have a spatially uniform solution of the form

\[
x_j(n) = x(n), \quad x(n+1) = f(x(n)).
\]

The steady state of this solution is chaotic if the corresponding Lyapunov exponent \(\rho_0\) defined by

\[
\rho_0 = \lim_{n \to \infty} \left( \prod_{i=1}^{n} |f'(x_0(i))| \right)^{1/n}
\]

is greater than 1.

Models of CML have been studied, mostly with symmetric forces, i.e., \(\gamma_1 = \gamma_2\), and periodic boundary conditions \([1,2,4,5]\). Under these conditions it is straightforward to show that the spatially uniform state is unstable in the chaotic regime, i.e., for \(\rho_0 > 1\). In this paper we consider the asymmetric case \(\gamma_1 \neq \gamma_2\). We show that depending on the values of \(\gamma_1\) and \(\gamma_2\) the nature of the local instability may be of the convective type [9,10]. Hence it is very sensitive to the imposed boundary conditions. In particular, for open boundary conditions this coherent state will be stable in a nonmoving frame. Furthermore, we will show by numerical simulations that for these boundary conditions the uniform state is globally stable.

To study the stability of the uniform state we linearize Eqs. (1) about the uniform solution. To linear order in the deviations \(\xi_j(n) = x_j(n) - x(n)\), Eqs. (1) are

\[
\xi_j(n+1) = f'(x_0(n))(1 - \gamma_1 - \gamma_2)\xi_j(n) + \gamma_1 \xi_{j-1}(n) + \gamma_2 \xi_{j+1}(n).
\]

First we consider periodic boundary conditions \(\xi_j(n) = \xi_{j+N}(n)\). Then for large \(n\) the solution of Eqs. (4) can be written as

\[
\xi_j(n) = \sum_{m=-\infty}^{\infty} C_m \lambda^m(k_m) e^{ik_m n},
\]

where \(k_m = 2\pi m/N\) and \(m = 0, \pm 1, \pm 2, \ldots\), and the eigenvalue spectrum \(\lambda(k)\) is given by

\[
\lambda(k) = \rho_0 [1 - \gamma_1 (1 - e^{-ik}) - \gamma_2 (1 - e^{ik})].
\]

Stability of a mode implies that \(|\lambda(k)| < 1\). We are interested in the limit of a large system, \(N \to \infty\), with fixed \(\gamma_1\) and \(\gamma_2\). In this case, the spectrum \(\lambda_m\) given by Eq. (6) forms a continuum band, and in particular \(\lambda(k) \to \rho_0\) for \(k \to 0\). Hence if \(\rho_0 > 0\) not only is the uniform mode, \(m=0\), unstable but there is a band of nonuniform unstable modes, implying that under small perturbation the spatial coherence of the system will be destroyed.

To study the nature of this instability we consider the evolution of a smooth initial perturbation, i.e., we solve Eqs. (5) assuming \(C_m \to 0\) for \(m \to \infty\). Converting the sum over \(m\) to an integral over \(k\) and evaluating it in the long time limit using the saddle-point method, one finds that \(\xi_j(n) \approx \lambda^n(k_0)\). The complex wave number \(k_0\) is determined by the saddle-point condition \(\partial \ln \lambda(k)/\partial k = 0\) [11]. Using Eq. (6) yields

\[
\lambda(k_0) = \rho_0 [1 - \gamma_1 (1 - e^{-ik_0}) - \gamma_2 (1 - e^{ik_0})] = \rho_0 [1 - (\gamma_1/2 - \gamma_2/2)^2],
\]

For values of \(\gamma_1\) and \(\gamma_2\) such that \(\lambda(k_0) < 1\) the instability is convective: At each point \(j\), \(\xi_j(n)\) decays exponentially with \(n\) for large \(n\), although the perturbation grows with time in an appropriate moving frame.

The prediction regarding the decay of the perturbation
at any point ignored the boundary effects. In a system with periodic boundary conditions even convective instability will eventually lead to a destruction of the uniform state because a moving growing perturbation will always return to the origin. However, for open boundary conditions convective instability will not destroy the uniform state, as is shown below.

In the open boundary conditions, Eqs. (1) hold for all $j$ except for $j = 1$ and $N$ where the terms proportional to $\gamma_1$ and $\gamma_2$, respectively, are absent:

$$ x_i(n+1) = (1 - \gamma_2)f(x_1(n)) + \gamma_2 f(x_2(n)). $$

$$ x_N(n+1) = (1 - \gamma_1)f(x_N(n)) + \gamma_1 f(x_{N-1}(n)). $$

The same holds for the linearized equations, Eqs. (4). The uniform solution, Eq. (2), still exists and is unstable to uniform fluctuations, the eigenvalue of which is again $\rho_0 > 1$. However, the remaining $N - 1$ nonuniform eigenvalues of Eqs. (5) change. They are given by

$$ \xi_j(n) = \rho^m(k_m) e^{0} \cos(k_m + \varphi_m). $$

where $k_m = \tan^2(j/\gamma_2)$, $m = 1, 2, \ldots, N - 1$, $\varphi_m = \tan^{-1}(\gamma_1/\gamma_2)$, and $\tan^2(k_m) = [1 - (\gamma_1/\gamma_2)^{1/2} \cos(k_m)]/\sin(k_m)$. The eigenvalue spectrum of these modes is

$$ \rho(k) = \rho_0 [1 - \gamma_1 - \gamma_2 + 2(\gamma_1/\gamma_2)^{1/2} \cos k]. $$

For $\gamma_1 < \gamma_2$ this spectrum possesses a gap at $k = 0$, since $\rho(k = 0) - \lambda(k_0)$, of Eq. (7), which is less than the $m = 0$ eigenvalue $\rho_0$. In particular in the convective regime, i.e., for $\gamma_1$ and $\gamma_2$ such that $\lambda(k_0) < 1$, all the $N - 1$ nonuniform fluctuations are stable. The only instability which is left is the instability to uniform fluctuations which is inherent in the chaotic nature of the system. This instability, however, will not destroy the spatial coherence of the system.

The above predictions have been confirmed in numerical simulations of Eqs. (1) with open boundary conditions. We have used the logistic map $f(x) = \epsilon - x^2$ with $\epsilon = 1.67$ which is in the chaotic regime $\rho_0 = 1.26$. Figure 1 shows the temporal evolution of the system starting from randomly nonuniform initial conditions. The coupling constants are $\gamma_1 = 0.7$ and $\gamma_2 = 0.1$ for which $\lambda(k_0) < 1$. As is demonstrated in the figure, the left edge is a source of a synchronizing front that propagates with a finite velocity and leaves behind it a completely synchronized regime. The velocity of the front equals $\gamma_1 - \gamma_2$ (see, e.g., [12]). Thus, the basin of attraction of the uniform state is big, and probably covers most of the space of initial conditions.

Figure 1 shows that the synchronizing front does not propagate throughout the system. In the sufficiently long lattice it stops before reaching the right edge, creating a domain of synchronized state with a finite coherence length, $l_c$. This phenomenon is the result of the numerical noise. Indeed, it is found numerically that $l_c$ increases with the numerical precision, as demonstrated in Fig. 2.

![Figure 1](image1.png)

**FIG. 1.** Iterations of Eqs. (1) with $f(x) = 1.67 - x^2$, $\gamma_1 = 0.7$, $\gamma_2 = 0.1$, and $N = 100$, starting from 10, randomly chosen initial conditions. We show the results after $n = 1000$. The results reveal a coherent domain of length $l_c = 55$. The coherence length was defined from the condition $\max_{j \leq j \leq l_c} |x_j(n) - x_i(n)| \leq 10^{-8}$.

The sensitivity to the numerical noise indicates that the synchronized state is unstable to dynamic local noise. It is indeed expected that a weak noise will result in a finite coherence length. The reason is that each of the chaotic maps responds strongly to the perturbation of the noise. Maps near the left edge remain strongly synchronized with the map at the edge since the "synchronizing force" from the left maps is stronger than the noise. However, as we move along the chain the desynchronization caused by the noise increases and is completely destroyed at $l_c$. As will be calculated below, in the case of weak noise, $l_c \propto \ln \sigma$, where $\sigma$ denotes the amplitude of the noise. On the other hand, the width of the domain wall separating the synchronized and the completely unsynchronized

![Figure 2](image2.png)

**FIG. 2.** The dependence of $l_c$ on $\gamma_2$. All other parameters are the same as for Fig. 1. The data points represent estimates of $l_c$ based on the results of numerical simulations of Eqs. (1), with $\gamma_2$ varying between 0.01 and 0.15. The upper part corresponds to simulations with a quadratic precision (32 digits after decimal point), and the lower to double precision (16 digits). Solid lines represent the best fit by Eq. (15) with $\sigma_0 = 1 - \lambda(k_0)$ and $\sigma$ a fitting parameter.
parts of the lattice is rather sharp. It is of order a few lattice constants.

To estimate the effect of a weak noise, we add to the right-hand side (RHS) of Eqs. (1) terms \( \eta_j(n) \) representing a white uniformly distributed noise with a width \( \sigma \). Assuming \( \sigma \ll 1 \) we evaluate the perturbation of the uniform state, Eq. (2), in the linear regime, using Eqs. (4) with the noise \( \eta_j(n) \) added to its RHS. The coherence is defined as the minimal length \( l \) for which \( \xi^2_j(n \to \infty) \approx 1 \). Solving Eqs. (4) with the added noise we find

\[
(\xi^2_j(n \to \infty)) = \sigma^2 \sum_{m_1=1}^{N} \sum_{m_2=1}^{N} \psi_m(j) \psi_m(j) \frac{Z(m_1,m_2)}{1 - \rho_{m_1} \rho_{m_2}},
\]

(10)

where

\[
Z(m_1,m_2) = \sum_{j=1}^{N} \psi^*_m(j) \psi^*_m(j).
\]

Finally, equating Eq. (10) to 1 yields

\[
\text{l}_c = \frac{\ln(\sigma / \sigma_0)}{\delta + \kappa}.
\]

(15)

The constant \( \sigma_0 \) is roughly equal to \( 1/\ln[1 - \rho(k \to 0)] \). Near the critical point it diverges, causing the vanishing of \( l_c \). The dependence of \( l_c \) on \( \ln \sigma \) is caused by the spatial amplification of the small perturbations due to the convective instability. The effect of noise on spatial coherence has also been discussed in [3,5].

The analytical result for \( l_c \) agrees well with the numerical simulations, as shown in Fig. 2. In the present case \( l_c \) drops to zero at \( \gamma_2 \approx 0.138 \) where \( \rho(k \to 0) = \lambda(k_0) \to 1 \). Note, however, that the vanishing of \( l_c \) is relatively sharp in agreement with the predicted logarithmic singularity.

In conclusion, we have shown that a CML with sufficiently asymmetric short-range couplings and open boundary conditions can have a stable spatially uniform but temporally chaotic state. The signature of this state is a big gap, near \( k = 0 \), in the spectrum of Lyapunov exponents of the linear operator characterizing the evolution of the fluctuations. These results extend those previously obtained for the purely unidirectional chain \( \gamma_2 = 0 \) [7]. This is a simple case in that a localized perturbation propagates with a finite velocity to the right and does not affect at all the maps behind it. In the present case where \( \gamma_2 \neq 0 \) the perturbation also diffuse backwards. Nevertheless, we have shown here that if the perturbing front propagates sufficiently fast its tail in the backward direction is exponentially small and may not be able to destabilize the maps despite their exponential sensitivity to perturbations.

It is straightforward to generalize our results to CML in higher dimensions. An interesting question is whether, in general, either the asymmetry in the interactions or the open boundary conditions are necessary prerequisites for the emergence of long-range-ordered chaotic states. We believe that this phenomenon may exist also in systems with symmetric interactions, since the underlying dynamical state can spontaneously break the symmetry and give rise to convective instabilities. Furthermore, it is not inconceivable that in such cases the long-range spatial order may be stable even with periodic boundary conditions. This will happen if the propagating perturbation fronts and those reflected from the boundaries will annihilate each other, leaving intact the spatial order. A similar mechanism of stabilizing spiral states was recently shown for the complex Ginzburg-Landau model [13]. These questions are under current investigation.

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