Solvable Model of Spatiotemporal Chaos

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A continuous time dynamic model of a d-dimensional lattice of coupled localized m-component chaotic elements is solved exactly in the limit $m \to \infty$. A self-consistent nonlinear partial differential equation for the correlations in space and time is derived. Near the onset of spatiotemporal disorder there are solutions that exhibit a novel space-time symmetry: the corresponding correlations are invariant to rotations in the $d+1$ space-time variables. For $d < 3$ the correlations decay exponentially at large distances or long times. For $d \geq 3$ the correlations exhibit a power law decay as the inverse of the distance or time.

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Many dynamical systems with a large number of degrees of freedom display a transition to a state that is highly irregular in both space and time as an appropriate control parameter is changed. An important question is the nature of the onset of spatiotemporal disorder in spatially extended systems. Another fundamental issue is the statistical characterization of the disordered state [1,2]. To gain insight into these questions, simple extended models have been studied, mostly numerically. They include lattices of coupled maps [3–5] and complex Ginzburg-Landau systems [6,7].

In this paper we introduce a solvable deterministic model that exhibits a transition from a fixed-point state to a state with spatiotemporal fluctuations. The model consists of a d-dimensional cubic lattice on each site of which there are m dynamical variables. The local variables are coupled nonlinearly and can potentially display local temporal chaos. The variables on different lattice sites are coupled by short-range interactions. The statistical properties of the system are studied in the limit of large m. In this limit, the fluctuating fields obey Gaussian statistics, with a correlation function in space and time that is determined self-consistently. We have solved the self-consistent equations for the correlations near the onset of the spatiotemporal disorder. Interestingly, in this regime the equations are invariant under space-time rotations. This symmetry reflects the strong coupling between spatial and temporal fluctuations in a deterministic extended system. For $d < 3$, there are many isotropic self-consistent solutions, all of which decay exponentially at large time and large distances, but they differ in the number of nodes they possess. Numerical simulations of the model at $d = 1$ are in a good agreement with the space-time isotropic nodeless solution. For $d \geq 3$ there are no (space-time isotropic) exponentially decaying solutions. Instead we find an isotropic solution in which the correlation function decays asymptotically as the inverse of the distance and time.

We consider a d-dimensional lattice, containing at each site $\mathbf{r}$, m real-valued variables $h_l(\mathbf{r}, t)$, $l = 1, ..., m$. They obey the following first order dynamic equations:

$$\frac{\partial h_l(\mathbf{r},t)}{\partial t} = -h_l(\mathbf{r},t) + \sum_{k=1}^{m} J_{lk}\, S(h_k(\mathbf{r},t)) + \frac{K}{2d} \sum_{\mathbf{a}} h_l(\mathbf{r}+\mathbf{a},t). \quad (1)$$

The second term of the right-hand side (rhs) of Eq. (1) represents the local nonlinearity. For concreteness we choose $S(h_l(\mathbf{r},t)) = \tanh(g h_l(\mathbf{r},t))$. The gain parameter g, $0 < g < \infty$, represents the degree of nonlinearity of the system. The local coupling matrix $\mathbf{J}$ is an asymmetric real random matrix drawn from a Gaussian distribution with zero mean and the following second moments: $[J_{lk} J_{l'k'}] = \delta_{ll'}\delta_{kk'}/m$. The square brackets denote average with respect to the Gaussian distribution of $\mathbf{J}$. The last term in Eq. (1) represents a nearest-neighbor linear interaction, with a coupling strength denoted by K, $0 < K < 1$. The vectors $\mathbf{a}$ are the lattice unit-cell vectors. It is important to emphasize that the matrix $\mathbf{J}$ is a global random matrix. Its value is the same at all sites $\mathbf{r}$. Thus, assuming periodic boundary conditions, the system is translationally invariant, unlike usual systems with quenched disorder.

The model of Eq. (1) was inspired by neural network dynamics. It corresponds to a network consisting of clusters of neurons with local excitatory and inhibitory coupling (given by J_{lk}) and excitatory interactions between the clusters (given by K). For $K = 0$, the local m-component fields follow the same dynamics as in the neural network model studied in [8]. In particular, they exhibit a transition at $g = 1$ from a zero fixed point to temporal chaos.

Because of the random sign of J_{lk}, the quantities $z_l(\mathbf{r},t) = \sum_k J_{lk}\, S(h_k(\mathbf{r},t))$ consist, in the limit $m \to \infty$, of sums of a large number of random variables and can therefore be replaced by Gaussian variables. Thus, in the large m limit, Eq. (1) can be reduced to a set of linear inhomogeneous equations:

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$$\frac{\partial h_l(\mathbf{r},t)}{\partial t} = -h_l(\mathbf{r},t) + \frac{K}{2d}\sum_{\mathbf{a}} h_l(\mathbf{r}+\mathbf{a},t) + z_l(\mathbf{r},t) . \quad (2)$$

The term $z_l(\mathbf{r},t)$ is a Gaussian noise with zero mean and correlations given by

$$\langle z_l(\mathbf{r},t) z_l(\mathbf{r}',t')\rangle = \langle S(h_l(\mathbf{r},t)) S(h_l(\mathbf{r}',t'))\rangle . \quad (3)$$

Different components of $\{z_l\}$ are uncorrelated. The averages in the rhs of Eq. (3) have to be calculated by solving for $h_l(\mathbf{r},t)$ as functionals of $z_l(\mathbf{r},t)$ via Eq. (2). Physically, averaging over the noise z_l is equivalent to averaging over time and space. Using a functional integral representation of Eqs. (1) it can be shown [9] that their reduction to the Gaussian model of Eqs. (2) and (3) is exact in the large m limit.

It is convenient to formulate the problem in terms of the correlation function $C(\mathbf{r},\tau) = \langle h_l(\mathbf{r_0},t) h_l(\mathbf{r_0}+\mathbf{r},t+\tau)\rangle$ where the average is over t and $\mathbf{r_0}$. By Fourier transforming Eq. (2) in space and time, squaring it, and expressing the rhs of Eq. (3) as a functional of C, we have derived the following self-consistent equation for C:

$$\frac{d^2C(\mathbf{r},t)}{dt^2} - \sum_{\mathbf{r}'} M(\mathbf{r},\mathbf{r}')C(\mathbf{r}',t) = -\frac{\partial V(C(\mathbf{r},t))}{\partial C(\mathbf{r},t)} . \quad (4)$$

The matrix M is given by $\mathsf{M} = (\mathsf{I}-\mathsf{K})^2 - (1-K)^2\mathsf{I}$, where K is the nearest-neighbor matrix $K(\mathbf{r},\mathbf{r}') \equiv (K/2d)\sum_{\mathbf{a}} \delta_{\mathbf{r}-\mathbf{r}',\mathbf{a}}$ and I is the identity matrix. The function $V(C(\mathbf{r},t))$ is given by

$$V(C) = -\frac{1}{2}(1-K)^2C^2 + \int_{-\infty}^{\infty} Dz \left[\int_{-\infty}^{\infty} Dx\, \Phi\left(x\sqrt{C(\mathbf{0},0)-|C|} + z\sqrt{|C|}\right)\right]^2 , \quad (5)$$

where $Dx = \frac{dx}{\sqrt{2\pi}}e^{-x^2/2}$ and $\Phi(x) = g^{-1}\ln\cosh(gx)$. Note that $V(C(\mathbf{r},t))$ is a self-consistent potential. It depends parametrically on the equal-time equal-space correlation $C(\mathbf{0},0)$. The physical solutions of Eq. (4) must obey the following important boundary conditions: (i) $C(\mathbf{0},0)$ is finite, (ii) $C(-\mathbf{r},-t) = C(\mathbf{r},t)$ [hence $\dot{C}(\mathbf{0},0) = 0$], (iii) $|C(\mathbf{r},t)| \le C(\mathbf{0},0)$, and (iv) $C \to 0$ when $|\mathbf{r}| \to \infty$ or $|t| \to \infty$. Solutions that do not decay to zero can be shown to be unstable.

Equation (4) has a simple solution $C(\mathbf{r},t) = 0$ for all $\mathbf{r}$ and t which corresponds to the fixed point $h_l(\mathbf{r},t) = 0$ of Eq. (1). Linearizing the dynamics, it can be shown that the zero fixed point is stable for $g < (1-K)$ or equivalently for $\epsilon < 0$, where the control parameter ϵ is defined as $\epsilon \equiv [g^2 - (1-K)^2]/2$. For $\epsilon > 0$ this state is unstable and the system is characterized by a nonzero solution of Eq. (4). Here we limit ourselves to the study of this equation for $\epsilon \ll 1$. In this regime, $C \ll 1$ and for the behavior at distances large compared to the lattice constant a one can perform a gradient expansion to the operator $M(\mathbf{r})$ in Eq. (4) and expand V in powers of C and $C(\mathbf{0},0)$. This yields the following partial differential equation:

$$\nabla^2_{d+1} C = -\frac{\partial V(C)}{\partial C}, \quad (6)$$

where ∇^2_{d+1} is the Laplacian in the $d+1$ space-time variables, i.e., $\nabla^2_{d+1} = \partial^2/\partial t^2 + \nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{r}}$, and

$$V(C) \approx -\frac{\sigma^2}{2}C^2 + \frac{\rho^2}{4}C^4, \quad (7)$$

where

$$\sigma^2 = -2\epsilon + 2g^4C(\mathbf{0},0) - 5g^6C^2(\mathbf{0},0) \quad (8)$$

and $\rho \equiv \sqrt{2g^6/3}$. In writing Eq. (6) we have rescaled $\mathbf{r}$ by $\mathbf{r} \to \mathbf{r}/v_0$ where $v_0 = a\sqrt{K(1-K)/d}$ is the microscopic velocity. Note that only the parameter σ vanishes at the onset.

Equations (6)–(8) are the fundamental mean-field equations. To determine the correlation function one must solve Eqs. (6) and (7) with the appropriate boundary conditions for fixed σ. To close the solution one uses Eq. (8) to determine σ as a function of ϵ. Equation (6) is invariant to rotations in space-time. This suggests considering solutions that have the same symmetry, namely,

$$C(\mathbf{r},t) = C(R), \quad R = \sqrt{\mathbf{r}^2 + t^2} . \quad (9)$$

Such solutions describe a state in which the temporal and spatial fluctuations are statistically indistinguishable. In most of the following we will focus on space-time isotropic solutions. Other solutions will be briefly discussed.

$d < 3$.—Near the onset of chaos $\sigma^2 \ll 1$ and Eq. (6) can be further simplified by defining $C(R) = (\sigma/\rho)u(\sigma R)$ yielding

$$u'' + \frac{d}{x}u' - u + u^3 = 0 , \quad (10)$$

where $x = \sigma R$. Nonlinear equations of the form of Eq. (10) in $d+1 = 2,3$ have been studied analytically and numerically mainly in the context of self-localized electromagnetic beams in two- and three-dimensional nonlinear media [10,11]. We are interested in solutions with a finite $u(0)$ and $u'(0) = 0$ that decay to zero at infinity. Such solutions behave asymptotically as $u(x) \sim x^{-d/2}\exp(-x), x \to \infty$, as can be seen from the linear part of Eq. (10). It can be shown that for $d = 1$ and 2 there is a countable set of exponentially localized solutions, $u_n(x)$. The index n denotes the number of nodes of $u_n(x)$. The values of the solutions at the origin, $u_n(0)$, are increasing with n. The simplest solution is the nodeless solution $u_0(x)$, which decreases monotonically with x. Each of these solutions generates a self-consistent solution for σ and hence for $C(R) = (\sigma/\rho)u_n(\sigma R)$. Taking into account that $\epsilon \ll 1$, Eq. (8) yields $\sigma \propto \epsilon$. This implies that near the onset of the spatiotemporal chaos the variance of the fluctuations behaves as

$$C(\mathbf{0},0) \propto \epsilon \ . \tag{11}$$

In addition, $C(\mathbf{r},t)$ decays exponentially both in space and in time with a correlation length ξ and a correlation time τ_c that behave as

$$\xi = \tau_c \propto \epsilon^{-1} \ . \tag{12}$$

In addition to isotropic solutions, Eqs. (6) and (7) possess families of anisotropic solutions [11]. These solutions decay exponentially with distance and time and yield the same critical behavior as that of the isotropic ones. In the present model the solutions with broken symmetry may correspond to states with damped traveling modes.

In order to test this theory, we have performed numerical simulations of Eq. (1) with a chain of 128 sites, each consisting of $m = 512$ variables. The equations were integrated using a forward Euler scheme ($dt = 0.1$). The correlation functions were computed by averaging over all sites and over a time period of 2048 steps after a transient. All the results are averages over 50 realizations of the random matrix $\mathbf{J}$. To reduce finite size effects (in m) we have chosen the parameters $g = 2, K = 0.2$, which are far from the onset. Results of the simulations are displayed in Fig. 1, where the values of $C(r,0)$ and $C(0,t)$ are presented vs $R = \sqrt{r^2/K(1-K)+t^2}$. The two sets of correlations are close to each other up to $R \approx 20$, suggesting that the continuum limit and the space-time symmetry remain approximately valid even far from the onset of chaos. To compare the simulation results with the quantitative predictions of the theory for $C(R)$, we have solved numerically Eq. (6) using a numerical evaluation of Eq. (5). The structure of the solutions remains the same as that for the approximate potential Eq. (7). The (space-time) isotropic, nodeless solution (Fig. 1, full line) is in a good quantitative agreement with the simulations of both $C(\mathbf{0},t)$ and $C(\mathbf{r},0)$. However, since the simulations are averages of different realizations of $\mathbf{J}$ we cannot rule out that for some initial conditions and realizations of $\mathbf{J}$ the system settles into other isotropic or anisotropic solutions. Thus, the relevance of the huge multiplicity of solutions (found for $d < 3$) remains to be studied.

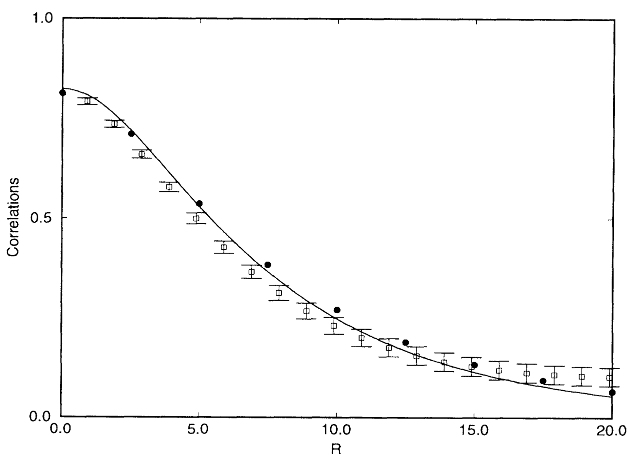

FIG. 1. Theory vs simulations for $g = 2, K = 0.2$. The correlation function $C(R)$, given by the isotropic, nodeless solution of Eq. (6), is shown by the full line. The squares are the results of the simulations of Eq. (1) for $C(0,t)$ plotted against $t = R$. The error bars show the standard deviation of the mean due to sample to sample fluctuations. The dots correspond to the simulation results for $C(r,0)$ plotted against $R = r\sqrt{K(1-K)}$. Here the errors are of the size of the symbols.

$d \geq 3$.—By numerical study of Eq. (10) we have found that for $d \geq 3$ this equation does not possess solutions that decay to zero at large distances and are regular at the origin. In fact, solving Eq. (10) with d as a real-valued parameter we find that the value of the nodeless solution at the origin diverges as $(3-d)^{-1}$ for $d \to 3^-$. This raises an important question regarding the nature of the dynamical state of our system at $d = 3$ and above. Here we propose that for $d \geq 3$ the correlation function satisfies Eqs. (6)–(8), but with

$$\sigma = 0, \quad \epsilon \geq 0 \ . \tag{13}$$

This means that above the onset the system is in a critical state, the consequences of which will become clear below. Substituting Eq. (13) in Eq. (6) and assuming as before an isotropic solution $C(R)$, we obtain

$$u'' + \frac{d}{x}u' + u^3 = 0 \ . \tag{14}$$

Here u was defined by the relation $C(R) = C(0)u(x)$ where $x = \rho C(0)R$, so that we look for a solution with $u(0) = 1$ [and $u'(0) = 0$]. The scale $C(0)$ is then determined by the self-consistent requirement of Eqs. (13) and (8), which yields $C(0) \propto \epsilon$. Note that in contrast to Eq. (10), Eq. (14) does not have a term proportional to u. It can be proven that for all $d > 3$ Eq. (14) has a one-parameter family of solutions that obey $u'(0) = 0$ and decay to zero at infinity. The decay is according to the power law

$$u(x) \propto 1/x, \quad x \gg 1/u(0) \ . \tag{15}$$

The solutions are parametrized by the free parameter $u(0)$, which in our case has to be equal to 1, as was stated above.

The case $d = 3$ is marginal. In this case Eq. (14) yields the following one-parameter family of solutions

$$u(x) = u(0)/\{1 + [xu(0)]^2/8\} \tag{16}$$

that are regular at the origin and behave asymptotically as a power law: $u(x) \sim 1/x^2$ for $x \gg 1/u(0)$. The value of $u(0)$ is arbitrary and can be chosen to be 1. However, this solution is unstable to small perturbations of the shape of the potential $V(C)$. Expanding Eq. (5) to the sixth power of C leads to the addition of a term proportional to $C^2(0)u^5$ to Eq. (14). Taking this term into account, we find that the solution for $d = 3$ crosses over from the behavior given by Eq. (16) at $x \gg 1/u(0)$ to

$$u(x) \approx F(\ln x)/x, \quad x \gg 1/(u(0)C(0)) \quad , \tag{17}$$

where $F(x)$ is a periodic function of x with a period proportional to $|\ln C(0)|$. Here too $x = \rho C(0)R$. Finally, as in the case of $d > 3$ the value of $C(0)$ is determined by Eqs. (8) and (13) yielding Eq. (11).

To conclude, for $d = 3$ and above the only self-consistent solution for the correlation function which is isotropic in space-time is a solution that decays as an inverse power law in both time and space. The system adjusts the value of the amplitude $C(0)$ of the spatiotemporal fluctuations so that it will remain in a critical state ($\sigma = 0$) even above the onset. The order parameter $C(0)$ also determines the scale in space and time above which the algebraic decay in the correlations sets in. This parameter vanishes continuously as the onset is approached from above, as $C(\mathbf{0}, 0) \propto \epsilon$. Whether there exist, at $d \geq 3$, anisotropic solutions that decay exponentially is an open question.

So far, we have focused on the properties of the space-time correlation function. Another characterization of the state is the spectrum of its Lyapunov exponents. Although it is hard to measure these exponents in an extended system, they are important theoretically in that they characterize the sensitivity of the system to perturbations of its initial conditions. Using a method similar to that of [8] we have studied analytically the maximal Lyapunov exponent λ_0 in our system near the onset $0 < \epsilon \ll 1$. For $d < 3$ we have found that λ_0 is positive and grows as $\lambda_0 \propto \epsilon^2$. Thus, $\xi \propto 1/\sqrt{\lambda_0}$ [see Eq. (12)]. A related scaling relation was found in a coupled maps lattice model [4]. We find a positive maximal Lyapunov exponent also for the algebraic states at $d \geq 3$. Thus, the predicted power-law phase at three dimensions and above suggests that critical phases may appear generically in temporally chaotic systems even in the absence of an external noise or conservation laws. This issue has been debated in the context of self-organized criticality [12].

The predicted space-time symmetry is incompatible with the conjecture [13] that the onset of spatiotemporal disorder belongs to the universality class of directed percolation (DP), which is characterized by critical space-time anisotropy. However, numerical and experimental tests of this conjecture have been inconclusive [14]. The analogy with DP relies on the assumed existence of a unique locally stable laminar state, a condition that does not hold in our model as well as in some of the other studied models. In conclusion, while for some systems and models the DP transition may be a probable candidate, our paper shows that there is another universality class for the transition to spatiotemporal chaos. Which physical systems and models realize these different behaviors is a matter of future studies. We hope that the present theory will motivate new numerical and experimental work that will further elucidate the nature of this transition.

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